Wiener—Brioschi approach to relativistic quantum mechanics

Yu. P. Rybakov
RUDN University
6, Mikluho-Maklay str., 117198 Moscow, Russia
E-mail: soliton4@mail.ru

Abstract

We consider the essence of the well-known discussion between Bohr and Einstein (1935) which concerned the completeness of quantum mechanics. If one followed Bohr, then the wave function would give the probability description of an individual particle. However, Einstein considered the wave function as an instrument for describing the statistical ensemble of identical particles-solitons. On the other hand, Wiener found the special α-representation of quantum mechanics for which the wave function appeared to be an element of the random Hilbert space with the normal dispersion. This fact proves the equivalence of Bohr and Einstein positions, the central limiting theorem being taken into account. Moreover, we show that within the scope of the Skyrme—Faddeev chiral model particles can be considered as self-gravitating solitons. Therefore, we infer that for island systems with the plane asymptotic space-time quantum mechanics ensues from Einstein gravity equations.

Introduction

Let us begin with an old discussion (1935) between Bohr and Einstein [1,2] concerning completeness of quantum mechanics. As well known, Bohr assumed that the wave function gave the probability description of an individual particle. According Einstein, the wave function described the quantum statistical ensemble of identical particles. Thus, the suggestion on probability description appeared to be common both for Bohr and Einstein. Moreover, Einstein suggested a tremendous program for geometrizing physics and considered quantum particles (electrons or photons) as pulsating solitons, that is clots of some fundamental field satisfying nonlinear equations.

Later on Wiener showed [3] the equivalence of these points of view. He constructed the special representation of quantum mechanics where the wave function appeared to be an element of a random Hilbert space with the Gaussian dispersion. Nowadays this representation is known as the stochastic integral by Wiener—Ito—Stratonowich. However, the approaches by Wiener and Einstein prove to be also equivalent [4] since the Wiener's wave function can be represented as the sum of large number of solitons with random phases. As well known, in accordance with the central limiting theorem [5], the latter sum behaves as a Gaussian random variable.

Brioschi spinors and topological solitons

Now it is worth while to stress the important contribution in realizing the Einstein's program made by the Italian geometrician Francesco Brioschi (1824–1897). He used complex projective coordinates (16-spinors) to study the 8-dimensional space geometry[6]. It can be shown that Brioschi 16-spinor solitons prove to be stable within the scope of the Skyrme—Faddeev chiral model(SFCM) [7,8]. This fact permits one to construct the wave functions of quantum particles considered as self-gravitating asymptotic solutions to field equations.

The equivalence of Bohr and Einstein approaches becomes evident if one introduces a soliton solution $u(t, \mathbf{r}) \equiv \mathbf{u}(\mathbf{x})$ to some nonlinear equation (with linear Klein—Gordon part) describing a massive extended particle. Let us also consider the de Broglie plane wave

$$\psi = A\exp[-\imath\omega t + \imath(\mathbf{kr})] \equiv \mathbf{A}\exp(-\imath\mathbf{kx}) \tag{1}$$

for a free particle with the energy ω, the momentum $\mathbf{k}$ and the mass m, where the relativistic relation $m^2 = k^2 = \omega^2 - \mathbf{k}^2$ determines the soliton's size $\ell_0 = m^{-1}$, with the natural units being $\hbar = c = 1$. It will be shown that the wave (1) can be represented as the sum of solitons located at the nodes $\mathbf{d}$ of a cubic lattice with the spacing $a >> \ell_0$:

$$\psi = \sum_{\mathbf{d}} u(t, \mathbf{r} + \mathbf{d}). \tag{2}$$

The result (2) follows from the asymptotic behavior of the soliton in its tail region:

$$u(x) = \int d^4k \, \exp(-\imath kx)\, g(k)\, \delta(k^2 - m^2). \tag{3}$$

Inserting (3) into (2) and using the well-known Poisson formula [9]:

$$\sum_{\mathbf{d}} \exp[\imath(\mathbf{kd})] = (\mathbf{2}\pi/\mathbf{a})^{\mathbf{3}}\, \delta(\mathbf{k}),$$

one derives from (1) that $A = (2\pi/a)^3 g(m)/(2m)$.

Wiener approach to quantum mechanics

As a result, the de Broglie wave (1) describes an ensemble of particles-solitons, though in idealized approximation, with the solitons' phases being matched and the lattice being cubic. However, in a realistic situation the solitons' phases appear to be random as well as the disposition of lattice nodes. Under these conditions, as well known, the central limiting theorem is valid [4,5]. It means that any large collection of solitons behaves as a Gaussian random quantity. Due to this fact, the wave function proves to be an element of the random Hilbert space with the scalar product $(\psi_1, \psi_2) = \mathbb{M}(\psi_1^* \psi_2)$, where the mathematical expectation $\mathbb{M}$ is taken over random parameters of solitons.

Precisely this interpretation of the wave function was suggested by Wiener [3] who considered a real Brownian process $x(s, \alpha)$ with the evolution parameter $s \in [0, 1]$ and the trajectory index $\alpha \in [0, 1]$, the latter one determining the correlation

$$\int_0^1 d\alpha x(s, \alpha)\, x(s', \alpha) = \min(s, s'). \tag{4}$$

For quantum description of a particle in $\mathbb{R}^3$ Wiener introduced the complex Brownian process

$$z(s|\alpha, \beta) = 2^{-1/2}[x(s, \alpha) + \imath y(s, \beta)]; \quad \alpha, \beta \in [0, 1],$$

and used the mapping $\mathbb{R}^3 \Rightarrow [0, 1]$ for constructing the stochastic representation of the wave function

$$(\alpha, \beta|\psi) = \int_{s \in [0,1]} dz(s|\alpha, \beta)\, \psi(s)$$

in the form of stochastic integral by Wiener—Ito—Stratonowich [10]. Moreover, Wiener proved, on the basis of the correlation (4), the unitarity of this representation transform:

$$\int_0^1 ds\, |\psi(s)|^2 = \iint d\alpha\, d\beta\, |(\alpha, \beta|\psi)|^2.$$

The SFCM and correspondence with quantum mechanics

Let us now show that within the scope of the SFCM in 16-spinor Brioschi realization the correspondence with quantum mechanics retains. To this end, let us consider the main part of the spinor Lagrangian density for the SFCM [7,8]:

$$\mathcal{L}_{\rm spin} = \frac{D}{2\lambda^2} + \frac{2D^3}{\lambda^2\, K^2 \varkappa_0^8\, \ell_{\rm Pl}^4} \left(J_\mu J^\mu - \varkappa_0^2\right)^2 , \tag{5}$$

where $J_\mu = \overline{\Psi}\gamma_\mu\Psi$, $D = \overline{\nabla_\mu\Psi}\gamma^\nu J_\nu \nabla^\mu\Psi$, $\nabla_\mu = \partial_\mu - \Gamma_\mu$, Γ_μ is the spinor affine connection, Ψ is the Brioschi 16-spinor, $K = R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau}/48$ is the so-called Kretschmann invariant constructed with the help of the Riemannian curvature tensor, and $\ell_{\rm Pl} = [c^3/(\hbar G)]^{1/2}$ is the Planckian length. Here it should be also included the gravitational Lagrangian coinciding with the Einstein's one.

Let us now suppose the existence of the soliton-like solution with the mass M and asymptotically plane space-time. It means that this soliton configuration behaves as an island-like excitation of the vacuum $\Psi_0 = \text{const}$, the condition of spontaneous symmetry breaking being fulfilled:

$$\Psi_0^+\Psi_0 = \varkappa_0.$$

Let us also consider a small perturbation ξ of the vacuum at large distances r from the soliton center. Therefore, at $r \to \infty$ one finds the development $\Psi = \Psi_0 + \xi$, $\xi \to 0$, that permits one to estimate Γ_μ and invariants in (5) through the Schwarzschild metric:

$$D = -\frac{r_{\rm g}^{*2}\,\varkappa_0^2}{4\,r^4}; \quad K = \frac{r_{\rm g}^{*2}}{r^6}; \quad r_{\rm g}^{*2} = \frac{GM}{c^2}. \tag{6}$$

Inserting (6) into (5) and using the substitution $\xi = k\,\Psi_0$, one gets the following linear equation for the real scalar multiplier $k = k^*$:

$$(\Box - m^2)\,k = 0, \tag{7}$$

where $m = Mc/\hbar$.

Conclusion

As follows from (7), the equation for the vacuum excitation ξ coincides with that of Klein—Gordon and also with the Schrödinger equation in the non-relativistic approximation. It means that the vacuum excitation ξ plays the role of the wave function in the special stochastic representation [8] It should be also underlined that the vacuum spinor Ψ_0 takes in general some complex values, the same being valid for the excitation ξ. In particular, for the substitution $\xi = \imath k\,\Psi_0$ one finds the wave equation $\Box\xi = 0$, describing massless perturbations of the vacuum (photons) [7]. Finally, it should be noticed that the proper gravitational field of particles plays an important role in our approach since the wave-particle duality principle of quantum mechanics has the gravitational origin.

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