Photon emission accompanying vacuum instability under the action of a quasi-constant electric field

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Abstract

Following a nonperturbative formulation of strong-field QED developed in our earlier works, we consider photon emission accompanying vacuum instability under the action of a quasi-constant strong electric field of finite duration T. We construct closed formulas for the total probabilities and study the photon emission by an electron and for the photon emission accompanying an electron-positron pair creation from a vacuum. We study angular and polarization distribution of the emission as well as emission characteristics in a high-frequency approximations with respect of $1/T$. It is a further development of the locally constant field approximation proposed in [Phys. Rev. D **95**, 076013 (2017)].

1 Introduction

Then in 1951 Schwinger found the vacuum-to-vacuum transition probability in a constant electric field. It became clear that the effect can actually be observed as soon as the external field strength approaches the characteristic value (critical field) $E_c = m^2c^3/|e|\hbar \simeq 1.3 \times 10^{18}\ V/m$, where $-e$ is the charge and m is the mass for an electron. Although a real possibility of creating such fields under laboratory conditions does not exist now, these fields can play a role in astrophysics, where the characteristic values of electromagnetic fields can be enormous. We consider the $3+1$ dimensional QED in the presence of the T-constant uniform electric field that exists during a macroscopic large time period T comparing to a characteristic time scale $\Delta t_{\mathrm{st}} = (e\,|E|\,c/\hbar)^{-1/2}$. This field turns on to E at $-T/2 = t_1$ and turns off to 0 at $T/2 = t_2$. Switching on

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and off effects of in the latter field can be neglected if we suppose that the time interval T is sufficiently large, namely

$$T/\Delta t_{st} > \max\{1, E_c/E\} \ . \tag{1}$$

We have the characteristic frequency $\omega_{sc} = \Delta t_{st}^{-1}$. The characteristic wavelength scale is $l_{sc} = 2\pi c \omega_{sc}^{-1}$.

Following a nonperturbative formulation of strong-field QED developed in [1], we consider high-frequency photon emission, $\omega \gg T^{-1}$, accompanying vacuum instability under the action of a quasi-constant strong electric field E of finite duration T, $E = aE_c$, where E_c is the critical field and a is a dimensionless parameter, $1 \gtrsim a \gtrsim 10^{-3}$. It is a further development of the locally constant field approximation proposed in [2]. In what follows, we use the relativistic units $\hbar = c = 1$ in which the fine structure constant is $\alpha = e^2/c\hbar = e^2$.

2 Effective perturbation theory of the photon emission

Radiative processes corresponding to the emission of a single photon might occur either from an electron (a positron) or from the vacuum accompanied by the creation of an electron-positron pair. The relative probability that a photon with momentum $\mathbf{k}$ and polarization ϑ is emitted from the vacuum accompanied by an electron-positron pair has the form :

$$\begin{aligned} &\mathcal{P}_1\left(\mathbf{k}\vartheta|0\right) = \sum_{n',n} \left| w^{(1)}\left(\overset{+}{n}{}'\overset{-}{n}; \mathbf{k}\vartheta|0\right)\right|^2 |c_v|^2 \ , \\ &w^{(1)}\left(\overset{+}{n}{}'\overset{-}{n}; \mathbf{k}\vartheta|0\right) = \left\langle 0, \mathrm{out}|a_{n'}\left(\mathrm{out}\right) b_n\left(\mathrm{out}\right) c_{\mathbf{k}\vartheta} S^{(1)}|0, \mathrm{in}\right\rangle c_v^{-1} \ . \end{aligned} \tag{2}$$

Here, the a's, b's, and c's are annihilation operators of electrons, positrons, and photons, respectively. Their adjoints are creation operators. The indices $n = (\mathbf{p}, \sigma)$ and $n' = (\mathbf{p}', \sigma')$ denote the complete set of particle's and antiparticle's quantum numbers, where $\mathbf{p}$ is momentum and $\sigma = \pm 1$ denotes the spin polarizations. Here $c_v = \langle 0, \mathrm{out}|0, \mathrm{in}\rangle$ is the vacuum to vacuum transition amplitude. The external field in the model is directed along the axis x and can be described by the vector potential with only one nonzero component,

$$A_x^{\mathrm{ext}}(t) = -Et \text{ if } t_1 \le t \le t_2.$$

It is assumed that for $t < t_1$ and for $t > t_2$, the electric field is absent, therefore initial $|0, \mathrm{in}\rangle$ and final $|0, \mathrm{out}\rangle$ are vacuum state of free in- and out- charged particles, respectively. These vacua are different due to a difference of initial and final values of external electromagnetic field potentials. During the time interval $t_2 - t_1 = T$, the Dirac field interacts with the external field. Moreover,

$S^{(1)}$ stands for the S-matrix truncated at first-order with respect to the radiative interaction,

$$S=\mathcal{T}\exp\left[-i\int\hat{\jmath}_{\mu}\left(x\right)\hat{A}^{\mu}\left(x\right)dx\right]\approx 1+S^{(1)}\,,$$
$$S^{(1)}=-i\int\hat{\jmath}_{\mu}\left(x\right)\hat{A}^{\mu}\left(x\right)dx\,,\quad dx=dtd\mathbf{r}\,,\quad d\mathbf{r}=dx^{1}dx^{2}dx^{3}\,,\tag{3}$$

wherein $\mathcal{T}$ denotes the time-ordering operator, $\hat{\jmath}^{\mu}\left(x\right)=-\left(e/2\right)\left[\hat{\Psi}^{\dagger}\left(x\right)\gamma^{0}\gamma^{\mu},\hat{\Psi}\left(x\right)\right]$ is the current density field operator, γ^{μ} are Dirac's matrices, and the Dirac field operators $\hat{\Psi}\left(x\right)$, $\hat{\Psi}^{\dagger}\left(x\right)$, and the electromagnetic field operator $\hat{A}^{\mu}\left(x\right)$ are in the interaction representation. The Dirac field operators obey both the Dirac equation with the potential $\mathbf{A}^{\rm ext}(t)$.

The in- and out- operators of creation and annihilation of electrons and positrons are defined by the two representations of the quantum Dirac field $\hat{\Psi}\left(x\right)$ as

$$\begin{aligned}\hat{\Psi}\left(x\right) &= \sum_{n}\left[a_{n}\left(\text{in}\right)\ {}_{+}\psi_{n}\left(x\right)+b_{n}^{\dagger}\left(\text{in}\right)\ {}_{-}\psi_{n}\left(x\right)\right]\,,\\ &= \sum_{n}\left[a_{n}\left(\text{out}\right)\ {}^{+}\psi_{n}\left(x\right)+b_{n}^{\dagger}\left(\text{out}\right)\ {}^{-}\psi_{n}\left(x\right)\right]\,.\end{aligned}\tag{4}$$

where ${}_{\zeta}\psi_{n}\left(x\right)$ and ${}^{\zeta}\psi_{n}\left(t,\mathbf{r}\right)$ are normalized in- and out-solutions of the Dirac equation with the potential $\mathbf{A}^{\rm ext}(t)$ for well-defined sign of frequency ζ ($\zeta=+$ for electrons and $\zeta=-$ for positrons) either before turning on or after turning off of a field, respectively. They related by a linear transformation of the form:

$$\begin{aligned}{}^{\zeta}\psi_{n}\left(x\right) &= g_{n}({}_{+}|^{\zeta})\,{}_{+}\psi_{n}\left(x\right)+g_{n}({}_{-}|^{\zeta})\,{}_{-}\psi_{n}\left(x\right)\,,\\ {}_{\zeta}\psi_{n}\left(x\right) &= g_{n}\left({}^{+}|_{\zeta}\right)\,{}^{+}\psi_{n}\left(x\right)+g_{n}\left({}^{-}|_{\zeta}\right)\,{}^{-}\psi_{n}\left(x\right)\,,\end{aligned}\tag{5}$$

where the g's are some complex coefficients, $g\left({}^{\zeta'}|_{\zeta}\right)=g\left({}_{\zeta}|^{\zeta'}\right)^{*}$. These coefficients obey the unitarity relations. All the coefficients can be expressed in terms of two of them, e.g. of $g\left({}_{+}|^{+}\right)$ and $g\left({}_{-}|^{+}\right)$. However, even the latter coefficients are not completely independent,

$$\left|g_{n}\left({}_{-}|^{+}\right)\right|^{2}+\left|g_{n}\left({}_{+}|^{+}\right)\right|^{2}=1.\tag{6}$$

Then a linear canonical transformation (Bogolubov transformation) between in- and out- operators which follows from Eq. (4) is defined by these coefficients

$$\begin{aligned}a_{n}\left(\text{out}\right)&=g_{n}\left({}^{+}|_{+}\right)a_{n}(\text{in})+g_{n}\left({}^{+}|_{-}\right)b_{n}^{\dagger}(\text{in}),\\ b_{n}^{\dagger}\left(\text{out}\right)&=g_{n}\left({}^{-}|_{+}\right)a_{n}(\text{in})+g_{n}\left({}^{-}|_{-}\right)b_{n}^{\dagger}(\text{in}).\end{aligned}\tag{7}$$

Using relations (7), one finds that the differential mean number of the pairs created is

$$N_{n}^{\rm cr}=\left|g_{n}\left({}_{-}|^{+}\right)\right|^{2}.\tag{8}$$

The decomposition of the operator $\hat{A}\left(x\right)$ in terms of creation and annihilation operators of free photons, $c^{\dagger}_{\mathbf{k}\vartheta}$ and $c_{\mathbf{k}\vartheta}$, reads:

$$\hat{\mathbf{A}}(x)=c\sum_{\mathbf{k},\vartheta}\sqrt{\frac{2\pi}{V\omega}}\boldsymbol{\epsilon}_{\mathbf{k}\vartheta}\left[c_{\mathbf{k}\vartheta}\,e^{i(\mathbf{kr}-\omega t)}+c^{\dagger}_{\mathbf{k}\vartheta}\,e^{-i(\mathbf{kr}-\omega t)}\right], \tag{9}$$

where $\vartheta=1,2$ denotes a polarization index and $\epsilon_{\mathbf{k}\vartheta}$ are mutual orthogonal unit polarization vectors transversal to a wave vector $\mathbf{k}$.

The relative probability (2) describes the emission of a photon accompanied by a single electron-positron pair created from the vacuum. However, once the background field violates vacuum stability, this process can be followed by electron-positron pairs created from the vacuum. By summing these probabilities for all the pairs we obtain the total probability of one photon emission from the vacuum that can be represented as the following mean value of the photon number operator

$$\mathcal{P}\left(\mathbf{k}\vartheta|0\right)=\left\langle 0,\mathrm{in}\left|S^{-1}c^{\dagger}_{\mathbf{k}\vartheta}c_{\mathbf{k}\vartheta}S\right|0,\mathrm{in}\right\rangle, \tag{10}$$

where the S-matrix truncated at first-order, $S\approx 1+S^{(1)}$. The unitary transformation $\mathcal{V}$ relates the in and out- representations of the Fock space, $|\mathrm{in}\rangle=\mathcal{V}|\mathrm{out}\rangle$. It means that we can pass from the basis of the final states to the basis of the initial states and represent the total probabilty (10) as

$$\begin{aligned}
\mathcal{P}\left(\mathbf{k}\vartheta|0\right) &= \sum_{m,n}\left|w^{(1)}_{\mathrm{in}}\left(\overset{-}{m}\overset{+}{n};\mathbf{k}\vartheta|0\right)\right|^{2},\\
w^{(1)}_{\mathrm{in}}\left(\overset{-}{n}\overset{+}{n'};\mathbf{k}\vartheta|0\right) &= i\sqrt{\frac{2\pi}{V\omega}}\int e^{i(\omega t-\mathbf{kr})}\mathbf{j}_{\mathrm{in}}\left(\overset{-}{n}\overset{+}{n'}|0\right)\boldsymbol{\epsilon}_{\mathbf{k}\vartheta}dx\,,\\
\mathbf{j}_{\mathrm{in}}\left(\overset{-}{n}\overset{+}{n'}|0\right) &= \left\langle 0,\mathrm{in}\left|b_{n}\left(\mathrm{in}\right)a_{n'}\left(\mathrm{in}\right):\mathbf{j}\left(x\right):\right|0,\mathrm{in}\right\rangle.
\end{aligned} \tag{11}$$

Note that in the frequency range $\omega\gg T^{-1}$ the contribution due to the mean current $\left\langle 0,\mathrm{in}\left|\hat{\jmath}(x)\right|0,\mathrm{in}\right\rangle$ is neglected.

The electric field acting during the time T creates a considerable number of pairs only in a finite range in the momentum space where this number is identical with that of the constant electric field, $N^{\mathrm{cr}}_{n}\simeq e^{-\pi\lambda}$; see [3] for details. We call it as the range of stabilization for a creation process. In this range solutions of Dirac equation can be presented as

$$\begin{aligned}
{}_{\pm}\psi_{n}\left(x\right) &= \exp\left(i\mathbf{pr}\right)\,{}_{\pm}\psi_{n}\left(t\right),\ {}_{\pm}\psi_{n}\left(t\right)=\left\{\sqrt{eE}\left(\pm1+i\right)\,{}_{\pm}\varphi_{n,\mp1}(t)\right.\\
&\quad\left.+\gamma^{0}\left(\gamma^{2}p_{y}+\gamma^{3}p_{z}+m\right)\,{}_{\pm}\varphi_{n,\pm1}(t)\right\}v_{\chi,\sigma},\\
{}_{+}\varphi_{n,\chi}(t) &= CD_{\nu-(1+\chi)/2}\left[-(1-i)\xi\right],\ \ {}_{-}\varphi_{n,\chi}(t)=CD_{-\nu-(1-\chi)/2}\left[-(1+i)\xi\right],\\
\xi &= \frac{qEt-p_{x}}{\sqrt{eE}},\ \nu=i\lambda/2,\ \lambda=\frac{p_{y}^{2}+p_{z}^{2}+m^{2}}{eE},\ C=\frac{\exp(-\pi\lambda/8)}{\sqrt{2eEV}}.
\end{aligned} \tag{12}$$

where D's are the linearly independent Weber parabolic cylinder functions (WPCFs) [5], $v_{\chi,\sigma}$ is a set of constant orthonormalized spinors,

$$\gamma^0\gamma^1 v_{\chi,\sigma} = \chi v_{\chi,\sigma},\ \chi = \pm 1,\ i\gamma^2\gamma^3 v_{\chi,\sigma} = \sigma v_{\chi,\sigma},\ v^{\dagger}_{\chi,\sigma} v_{\chi',\sigma'} = \delta_{\chi,\chi'}\delta_{\sigma,\sigma'}\ .$$

Using the parametrization by frequency ω and solid angle $d\Omega$, $d\mathbf{k} = \omega^2 d\omega d\Omega$, one can write the probability of one photon emission with a given polarization ϑ per unit frequency and solid angle, which is accompanied by the pair production from the vacuum, as

$$\begin{aligned} \frac{d\mathcal{P}(\mathbf{k}\vartheta|0)}{d\omega d\Omega} &= \alpha \frac{\omega \Delta t_{st}^2}{(2\pi)^2} \sum_{\sigma'=\pm 1} \left| M^0_{n'n} \right|^2 \Big|_{\mathbf{p}'=\mathbf{p}-\mathbf{k}} \ , \\ M^0_{n'n} &= -\frac{V}{\Delta t_{st}} \int_{t_1}^{t_2} {}_{+}\bar{\psi}_{n'}(t)\gamma\boldsymbol{\epsilon}_{\mathbf{k}\vartheta}\ {}_{-}\psi_n(t) e^{i\omega t} dt \ , \end{aligned} \tag{13}$$

where representation (4) is used and integral over the space volume V is fulfilled.

Similarly to the previous process, the emission of a photon from an electron (a positron) can occur together with electron-positron pairs created from the vacuum. By summing these probabilities over the pairs we finally obtain the total probability of one photon emission from an electron (a positron) per unit frequency and solid angle that can be presented as

$$\begin{aligned} \frac{d\mathcal{P}\left(\mathbf{k}\vartheta|\overset{\pm}{n}\right)}{d\omega d\Omega} &= \alpha \frac{\omega \Delta t_{st}^2}{(2\pi)^2} \sum_{\sigma'=\pm 1} \left| M^{\pm}_{n'n} \right|^2 \Big|_{\mathbf{p}'=\mathbf{p}-\mathbf{k}} \ , \\ M^{\pm}_{n'n} &= \mp \frac{V}{\Delta t_{st}} \int_{t_1}^{t_2} {}_{\pm}\bar{\psi}_{n'}(t)\gamma\boldsymbol{\epsilon}_{\mathbf{k}\vartheta}\ {}_{\pm}\psi_n(t) e^{i\omega t} dt \ . \end{aligned} \tag{14}$$

In the range of stabilization, using the explicit representation (12) one can see that both $M^0_{n'n}$ and $M^{\pm}_{n'n}$ are linear combination of the following integrals

$$\begin{aligned} Y_{j'j} &= \Delta t_{st}^{-1} \int_{-\infty}^{\infty} D_{-\nu'-j'}\left[-(1+i)\xi'\right] D_{\nu-j}\left[-(1-i)\xi\right] e^{i\omega t} dt \ , \\ \tilde{Y}_{j'j} &= \Delta t_{st}^{-1} \int_{-\infty}^{\infty} D_{-\nu'-j'}\left[-(1+i)\xi'\right] D_{\nu-j}\left[-(1+i)\xi\right] e^{i\omega t} dt \ , \end{aligned} \tag{15}$$

where the limit $T \to \infty$ is used. It is taken into account that main contributions to the integrals (15) are formed in the neighborhood of the saddle-point

$$[2eEt - (p_x + p'_x)] = \omega. \tag{16}$$

In this neighborhood, the corresponding kernels have Gaussian forms with maxima at the time instant

$$t_c = \frac{1}{2}\left(\Delta t_{st}^2 \omega + \frac{p_x + p'_x}{eE}\right) \tag{17}$$

and with the standard deviation $\Delta t_{sd} = \Delta t_{st}/\sqrt{2}$. The time t_c corresponds to the position of the center of the formation interval Δt for given ω, p_x, and p_x'. The width of the formation interval Δt must be large enough to accommodate the points ξ' and ξ. The formation interval must overlap the standard deviation, $\Delta t_{sd} < \Delta t$. It is natural to assume that $\Delta t \sim \Delta t_{st}$. It implies that $\Delta t_{st} \left|k_x\right| < 1$.

3 High frequency approximation

Fourier transformations (15) can be expressed via the confluent hypergeometric function Ψ; see [4]. In the general case, angular and polarization distributions of the emitted photons have quite complicated form. Nevertheless, their analysis is greatly simplified in the range of high frequencies

$$\Delta t_{st}\omega \gg 1. \tag{18}$$

In this case, $\omega \gg |k_x|$ and integrals (15) can be approximated as

$$\begin{aligned}
&Y_{j'j} \approx (-1)^j \sqrt{\frac{2}{\pi}} \Gamma\left(\nu - j + 1\right) e^{i\pi\left(\nu' + j' - 1\right)/2} \sinh\frac{\pi\lambda}{2} I_{j',1-j}(\rho), &(19)\\
&\tilde{Y}_{j'j} = e^{i\pi\left(\nu + \nu' + j + j'\right)/2} I_{j',j}(\rho), &(20)\\
&I_{j',j}(\rho) = \sqrt{\pi}\exp\left[\left(\ln\frac{\rho}{\sqrt{2}} - \frac{i\pi}{4}\right)\left(\nu - \nu' + j - j'\right) + i\frac{\rho^2}{4} - \frac{i\pi}{4}\right] &\\
&\times\Psi\left(\nu + j, 1 + \nu - \nu' + j - j'; -i\frac{\rho^2}{2}\right), &(21)
\end{aligned}$$

where $\rho = \Delta t_{st}\omega$ and Γ is the gamma-function. Using an asymptotic behavior of the function Ψ given by Eq. (6.13.1.(1)) in Ref. [5], we find:

$$\Psi\left(\nu + j, 1 + \nu - \nu' + j - j'; -i\frac{\rho^2}{2}\right) = \left(-i\frac{\rho^2}{2}\right)^{-\nu - j}\left[1 + O\left(\rho^{-2(j+1)}\right)\right] .$$

We apply this theory to the calculation of total probability for the photon emissions in a constant electric field. We defined an orthonormal triple

$$\begin{aligned}
\mathbf{k}/k &= \left(\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta\sin\phi\right), \\
\boldsymbol{\epsilon}_{\mathbf{k}1} &= \mathbf{e}_x \times \mathbf{k}/\left|\mathbf{e}_x \times \mathbf{k}\right|, \quad \boldsymbol{\epsilon}_{\mathbf{k}2} = \mathbf{k}\times\boldsymbol{\epsilon}_{\mathbf{k}1}/\left|\mathbf{k}\times\boldsymbol{\epsilon}_{\mathbf{k}1}\right|
\end{aligned} \tag{22}$$

then

$$\begin{aligned}
\boldsymbol{\epsilon}_{\mathbf{k}1} &= \left(0,\, \sin\theta,\, -\cos\theta\right), \\
\boldsymbol{\epsilon}_{\mathbf{k}2} &= \left(\sin\eta,\, -\cos\eta,\, 0\right),\ \sin\eta = \frac{\sin\theta\tan\phi}{\sqrt{1 + \left(\sin\theta\tan\phi\right)^2}}
\end{aligned} \tag{23}$$

for $\mathbf{k}$ in the upper spatial region, $k_z \geq 0$. The leading contributions to the amplitudes $M_{n'n}^0$ and $M_{n'n}^+$ given by Eqs. (13) and (14) arise from the terms with $\tilde{Y}_{00}$ and Y_{01}, respectively.

We find the main contributions the probability densities (13) and (14) are

$$\begin{aligned}
\sum_{\sigma'=\pm1}\left|M_{n'n}^{0}\right|^{2}\Big|_{\mathbf{p}'=\mathbf{p}-\mathbf{k}} &\approx \tilde{f}\left(\lambda,\lambda'\right)\left(\epsilon_{\mathbf{k}\vartheta}^{1}\right)^{2},\ \tilde{f}\left(\lambda,\lambda'\right)=2\pi\, e^{-\pi\left(\lambda+\lambda'\right)}\Big|_{\mathbf{p}'=\mathbf{p}-\mathbf{k}};\\
\sum_{\sigma'=\pm1}\left|M_{n'n}^{+}\right|^{2}\Big|_{\mathbf{p}'=\mathbf{p}-\mathbf{k}} &\approx f\left(\lambda,\lambda'\right)\left[\left(\epsilon_{\mathbf{k}\vartheta}^{2}\right)^{2}+\left(\epsilon_{\mathbf{k}\vartheta}^{3}\right)^{2}\right],\\
f\left(\lambda,\lambda'\right) &= \frac{8\pi e^{-\pi\left(\lambda'+\lambda/2\right)}\sinh\left(\pi\lambda/2\right)}{\lambda^{2}\left[\left(\lambda/2\right)^{2}+1\right]}. \qquad (24)
\end{aligned}$$

We see that the emission accompanying the pair creation from vacuum is distinguished by the fact that it is linearly polarized along the direction on the external field. Its intensity is maximum for directions orthogonal to external field.

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