Application of Lagrangian mechanics to the analysis of particle dynamics in a gravitational field

V. B. Belayev

V. G. Khlopin Radium Institute, Saint-Petersburg, Russia

wbelayev@yandex.ru

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Introduction

- In the general theory of relativity (GR), the definition of the momenta of material and light-like particles moving in curvilinear space-time, and the forces acting on them, aims to find relativistic corrections to Newton's theory of gravitation for a weak gravitational field. If the second derivatives of the coordinates along the path [1-3] are considered as components of the 4-vector of the force acting on a material particle of a unit mass, then in the Newtonian limit, the value playing the role of the passive gravitational mass turns out to depend on the direction of motion of the particle [4]. The same is true for a photon, if the second derivatives of coordinates are identified with the 4-force by an affine parameter, as which the coordinate time is chosen.
- Another approach is the choice of the Lagrangian of the particle, the definition of generalized forces as its partial derivatives with respect to the coordinate in accordance with Lagrange mechanics. [5-8]. In GR, the physical velocities of particles are associated with the components of the contravariant 4-velocity vector. Therefore, the physical force is aligned with the upper index vector associated with the generalized force vector. The energy and momenta of particles are considered as the components of the contravariant 4-vector of energy-momentum, as is done in [1] for a particle moving in the Minkovsky space-time.
- In the Fock proof [9] of the light motion along geodesics, the time component of the covariant 4-velocity vector is taken as the Hamiltonian. Application of the variational principle of the energy stationary integral (PESI) to the motion of a light-like [5-8] particle in a gravitational field does not lead to a violation of the isotropy of the light path. In the generalized Fermat's principle [10], a variation of the integral of the time component of the 4-velocity vector is used and gives the trajectory of light movement that coincides with the geodesic.

Equations of Lagrange mechanics

In the general theory of relativity, a four-dimensional pseudo-Riemannian space-time with coordinates x^i and metric coefficients g_{ij} is considered, the interval in which is written in the form

$$ds^2 = g_{ij}dx^i dx^j . \tag{2.1}$$

The 4-velocity vector of the particle is denoted as $u^i = dx^i / d\mu$, where μ is the variable parameter. We obtain the equations of its dynamics in general form.

The particle Lagrangian corresponds to the covariant generalized momenta

$$p_i = \frac{\partial L}{\partial u^i} \tag{2.2}$$

and generalized forces

$$F_i = \frac{\partial L}{\partial x^i} . \tag{2.3}$$

The particle motion is determined by Hamilton's principle of stationary action $\delta S = 0$ at

$$S = \int_{\mu_0}^{\mu_1} L d\mu \ , \tag{2.4}$$

where μ_0, μ_1 are the values of the parameter at the points that are connected by the desired trajectory of motion. The extremum condition leads to the Euler-Lagrange equations

$$\frac{d}{d\mu}\frac{\partial L}{\partial u^\lambda} - \frac{\partial L}{\partial x^\lambda} = 0 . \tag{2.5}$$

Taking into account the expressions for generalized momenta (2.2) and forces (2.3), these equations are rewritten in the form

$$\frac{dp_\lambda}{d\mu} - F_\lambda = 0 . \tag{2.6}$$

Equations of Lagrange mechanics

The Lagrangian is chosen [5] so that contravariant momenta bind to the physical energy and momentum of the particle

$$p^j = g^{j\lambda} p_\lambda , \tag{2.7}$$

and the gravitational force acting on it is mapped to associated with (2.3) vector

$$F^l = g^{l\lambda} F_\lambda . \tag{2.8}$$

Passing to them in equations (2.6), we find

$$g_{\lambda i} F^i = g_{\lambda i} \frac{dp^i}{d\mu} + \frac{\partial g_{\lambda i}}{\partial x^l} u^l p^i . \tag{2.9}$$

Multiplying these equations by $g^{k\lambda}$ and summing over the twice occurring index λ, we obtain

$$F^k = \frac{dp^k}{d\mu} + g^{k\lambda} \frac{\partial g_{\lambda i}}{\partial x^l} u^l p^i . \tag{2.10}$$

The presence of the second term on the right side reflects that in the gravitational field not only the 4-momentum of matter, but the 4-momentum of matter together with the gravitational field is stored (see [1] § 96). Its components express the rate of change of the energy and momentum acquired by the gravitational field when a particle moves in it

$$\frac{d\vec{p}^k}{d\mu} = g^{k\lambda} \frac{\partial g_{\lambda i}}{\partial x^l} u^l p^i . \tag{2.11}$$

Equations of Lagrange mechanics

Integration of this quantity over gives the energy and momentum received by the gravitational field at a certain interval of its trajectory. As a result, equation (2.10) can be written in the form

$$F^k = \frac{dp^k}{d\mu} + \frac{d\vec{p}^k}{d\mu}. \tag{2.12}$$

It follows from the laws of conservation of energy and momentum that the force acting on a particle is equal in magnitude and opposite in sign to the force acting by the source of gravity from the side of the particle. This is equivalent to fulfilling Newton's third law.

Dynamics of the material particle

Let us consider the dynamics of a material particle [5]. The Lagrangian of particle with a rest mass m is as follows

$$L_m = cm\sqrt{g_{ij}u^i u^j}, \tag{3.1}$$

For material particles, the parameter μ coincides with the interval: $\mu = s$. The physical energy and momentum of the particle are associated with contravariant momenta (2.7), which take the form

$$p^i = cmu^i. \tag{3.2}$$

This choice is because only in this case the components of the momentum 4-vector coincide in sign with the components of the 4-velocity vector. The first component determines the energy of the particle

$$E = cp^1. \tag{3.3}$$

The gravitational force acting on a material particle, in view of (2.8), is determined by the formula

$$Q^k = cF^k = \frac{1}{2}c^2 m g^{k\lambda}\frac{\partial g_{ij}}{\partial x^\lambda}u^i u^j. \tag{3.4}$$

According to the general theory of relativity, the motion of a material particle is determined by the equations of the geodesic line. For a material particle with the Lagrangian (3.1), they can be obtained from Hamilton's principle of stationary action [9] and are identical to equations (2.6).

A material particle in the Schwarzschild field

The gravitational field of a spherical body outside its source in spherical coordinates is described by the Schwarzschild metric

$$ds^2 = c^2\left(1-\frac{\alpha}{r}\right)dt^2 - \left(1-\frac{\alpha}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{4.1}$$

where constant

$$\alpha = \frac{2\gamma M}{c^2} \tag{4.2}$$

The coordinate system is chosen so that the motion of the particle occurs in the plane $\theta = \frac{\pi}{2}$.

Equations (2.6) for the time-like interval with the Lagrangian (3.1) yields

$$\frac{cdt}{ds} = \eta\left(1-\frac{\alpha}{r}\right)^{-1}, \tag{4.3}$$

$$\frac{dr}{ds} = \pm\sqrt{\eta^2 - \left(1+\frac{A^2}{r^2}\right)\left(1-\frac{\alpha}{r}\right)}, \tag{4.4}$$

$$\frac{d\theta}{ds} = 0, \tag{4.5}$$

$$\frac{d\varphi}{ds} = \frac{A}{r^2}. \tag{4.6}$$

Dividing equation (4.4) by time velocity (4.3) gives

$$\dot{r} = \pm c\left(1-\frac{\alpha}{r}\right)\sqrt{1-\frac{1}{\eta^2}\left(1+\frac{A^2}{r^2}\right)\left(1-\frac{\alpha}{r}\right)}, \tag{4.7}$$

where η, A are constants.

A material particle in the Schwarzschild field

For world lines with unlimited r value η determined by radial velocity at infinity $\dot{r} = V$ and will be

$$\eta_1 = \left(1 - \frac{V^2}{c^2}\right)^{-1/2}. \tag{4.8}$$

If the trajectory of the free-moving particle is such that the radial coordinate has a finite extreme value r_{ext}, then equation (4.7), due to the condition $\dot{r}(r_{ext}) = 0$ has solution

$$\eta_2 = \left[\left(1 + \frac{A^2}{r_{ext}^2}\right)\left(1 - \frac{\alpha}{r_{ext}}\right)\right]^{1/2}. \tag{4.9}$$

For radially unbounded trajectories having an axis of symmetry, the following equality holds: $\eta_1 = \eta_2$.

Substituting the found components of the 4-velocity vector into the expression (3.4), we find the only non-zero component of the gravitational force vector acting on the material particle:

$$Q^2 = \frac{c^2 m\alpha}{r^2}\left(\frac{1}{2} - \frac{\eta^2 r}{r - \alpha}\right) + \frac{cA^2}{r^3}\left(1 - \frac{\alpha}{2r}\right). \tag{4.10}$$

With weak gravity, unlimited radial motion ($A = 0$, $\eta = \eta_1$) and $\alpha / r << V^2 / c^2$ it is reduced to

$$\tilde{Q}^2 = -\frac{c^2 m\alpha}{2r^2}\left(\frac{c^2 + V^2}{c^2 - V^2}\right). \tag{4.11}$$

A material particle in the Schwarzschild field

However, when considering non-radial movement ($A \neq 0$) to avoid the appearance of a fictitious force component caused by using a spherical coordinate system, it is necessary to use an isotropic shape of the Schwarzschild metric in rectangular coordinates (ct, x, y, z). It can be accessed using the transformation

$$r = \left(1 + \frac{\alpha}{4\bar{r}}\right)^2 \bar{r}\ , \tag{4.12}$$

$$x = \bar{r}\cos\theta\cos\varphi, \quad y = \bar{r}\cos\theta\sin\varphi, \quad z = \bar{r}\sin\theta\ , \tag{4.13}$$

which yields

$$ds^2 = c^2 \left(\frac{1 - \dfrac{\alpha}{4\bar{r}}}{1 + \dfrac{\alpha}{4\bar{r}}} \right)^2 dt^2 - \left(1 + \frac{\alpha}{4\bar{r}}\right)^4 (dx^2 + dy^2 + dz^2). \tag{4.14}$$

We will consider motion in a plane $z = 0$ and look for the force acting on a particle at a point with coordinates $(ct, x, 0, 0)$ corresponding to $\theta = \varphi = 0$ in a spherical frame of reference. Plane coordinate transformations

$$x = \bar{r}\cos\varphi, \qquad y = \bar{r}\sin\varphi \tag{4.15}$$

at the point under consideration correspond to nonzero spatial components of the 4-velocity vector in a rectangular coordinate system

$$u_r^2 = \frac{dx}{d\mu} = \frac{d\bar{r}}{d\mu}, \qquad u_r^3 = \frac{dy}{d\mu} = \frac{d\varphi}{d\mu}\bar{r} \tag{4.16}$$

at $\mu = s$ for a material particle. Transformation (4.12) implies

$$dr = \left(1 - \frac{\alpha^2}{16\bar{r}^2}\right) d\bar{r}. \tag{4.17}$$

A material particle in the Schwarzschild field

In view of the covariance of the geodesic equations, one can pass from their solutions for the Schwarzschild metric in spherical coordinates (4.3) - (4.7) to the solution for the metric (4.14) by making transformations (4.12) - (4.13) and (4.16). As a result, we find nonzero components of the 4-velocity vector:

$$u_r^1 = \eta \left(\frac{1+\dfrac{\alpha}{4\bar{r}}}{1-\dfrac{\alpha}{4\bar{r}}} \right)^2 , \tag{4.18}$$

$$u_r^2 = \pm \frac{1}{\left(1-\dfrac{\alpha^2}{16\bar{r}^2}\right)} \left[\eta^2 - \left(1 + \frac{A^2}{\bar{r}^2\left(1+\dfrac{\alpha}{4\bar{r}}\right)^4} \right) \left(\frac{1-\dfrac{\alpha}{4\bar{r}}}{1+\dfrac{\alpha}{4\bar{r}}} \right)^2 \right]^{1/2} , \tag{4.19}$$

$$u_r^3 = \frac{A}{\bar{r}\left(1+\dfrac{\alpha}{4\bar{r}}\right)^4} . \tag{4.20}$$

Substituting the obtained components of the 4-velocity vector into the expression for force (3.4) gives its only non-zero component

$$Q_{rect}^2 = -\frac{c^2 m\alpha}{2\bar{r}^2\left(1+\dfrac{\alpha}{4\bar{r}}\right)^3} \left(\eta^2 \left[\left(1-\frac{\alpha}{4\bar{r}}\right)^{-3} + \left(1-\frac{\alpha}{4\bar{r}}\right)^{-2} \right] - \left(1+\frac{\alpha}{4\bar{r}}\right)^{-2} \right). \tag{4.21}$$

This expression does not depend on the constant A, if the particle motion corresponding to constant η (4.12). In weak gravity, for $\alpha / r << V^2 / c^2$ and, component Q^2_{rest} coincides with the expression for the force in spherical coordinates during radial movement (4.15). It is Newton's law of gravity with a passive gravitational mass of a material particle

$$m_p^g = m\frac{c^2+V^2}{c^2-V^2}. \tag{4.26}$$

However, in the general case, due to the noncovariance of the force vector (3.4) during the transformations of coordinates (4.16), (4.17) in the formula for the force in the Schwarzschild field in spherical coordinates (4.14), the resulting expression does not coincide with (4.25) for the radial motion of the particle. As an example, we consider the gravitational force acting on a stationary material particle. The constants $A=0$, $\eta=\eta_2$ and the distance from the center $r=r_{ext}$ correspond to this case. The nonzero component of the force vector (4.14) takes the form

$$Q^2 = -\frac{c^2\alpha m}{2r^2}, \tag{4.27}$$

and the component (4.25) obtained for the metric in isotropic rectangular coordinates will be

$$Q^2_{rest} = -\frac{c^2 m\alpha\left(1+\dfrac{\alpha}{2\bar{r}}\right)}{2\bar{r}^2\left(1-\dfrac{\alpha}{4\bar{r}}\right)\left(1+\dfrac{\alpha}{4\bar{r}}\right)^6}. \tag{4.28}$$

Substitution of (4.16) in this expression excluding small quantities of order higher than α/r yields

$$Q^2_{rest} = -\frac{c^2 m\alpha}{2r^2}\left(1+\frac{\alpha}{4r}\right). \tag{4.29}$$

Therefore, the analogy with Newtonian gravity and the passive gravitational mass of a material particle can only be talked about in the limit of weak gravity.

A special case of a system consisting of two moving bodies

We study a system of two bodies A and B with the same mass M, which move in opposite directions in the coordinate frame $K' = (t', x', y', z')$ with the velocities v and $-v$. It is assumed that at the time $t' = 0$ the distance δr between them can be neglected to determine the gravity created by this system in the considered area.

In weak gravity, the metric (4.14) in an approximate form becomes the following:

$$ds^2 = c^2\left(1 - \frac{\alpha}{\overline{r}}\right)dt^2 - \left(1 + \frac{\alpha}{\overline{r}}\right)(dx^2 + dy^2 + dz^2). \tag{5.1}$$

We apply the Lorentz transformations

$$t = \frac{t' + \dfrac{\tilde{v}}{c^2}x'}{\sqrt{1 - \dfrac{\tilde{v}^2}{c^2}}},\quad x = \frac{x' + \tilde{v}t'}{\sqrt{1 - \dfrac{\tilde{v}^2}{c^2}}},\quad y = y',\quad z = z' \tag{5.2}$$

to it under the condition $\alpha / \delta r << \tilde{v}^2 / c^2$. This condition means that the distortions of space and time caused by the presence of the Lorentz factor will be an order of magnitude greater than that caused by gravity. Therefore, the influence of gravity on the Lorentz transformations in this case is insignificant and they can be applied to the metric (5.1). Transformation of coordinates at

$$\overline{r}' = \sqrt{\left(\frac{x' + \tilde{v}t'}{\sqrt{1 - \dfrac{\tilde{v}^2}{c^2}}}\right)^2 + y'^2 + z'^2} \tag{5.3}$$

yields

$$ds^2 = c^2\left(1 - \frac{c^2 + \tilde{v}^2}{c^2 - \tilde{v}^2}\frac{\alpha}{\overline{r}'}\right)dt'^2 - \frac{4c^2\tilde{v}}{c^2 - \tilde{v}^2}\frac{\alpha}{\overline{r}'}dt'dx' - \left(1 + \frac{c^2 + \tilde{v}^2}{c^2 - \tilde{v}^2}\frac{\alpha}{\overline{r}'}\right)dx'^2 - \left(1 + \frac{\alpha}{\overline{r}'}\right)(dy'^2 + dz'^2). \tag{5.4}$$

A special case of a system consisting of two moving bodies

In associated with bodies reference frames K_A, K_B the gravity of each of them separately is described in the corresponding frame by the metric (5.1). Let us pass from these coordinate systems to K', using the Lorentz transformations at

$$\tilde{v} = v \tag{5.5}$$

and

$$\tilde{v} = -v. \tag{5.6}$$

If we represent metric coefficients in the form $g_{ij} = \eta_{ij} + h_{ij}$, where η_{ij} correspond to the Minkovsky metric, then with weak gravity, the ratio $h_{ij} \approx \sum_n h_{ij}^n$ is performed for the total field created by n subsystems [1] with metric coefficients $g_{ij}^n = \eta_{ij} + h_{ij}^n$. Summing the fields obtained after substitutions (5.5) and (5.6) into the metric (5.4), we find path interval in the vicinity of $t' = 0$ in a two-body system

$$ds^2 = c^2\left(1 - \frac{c^2+v^2}{c^2-v^2}\frac{\alpha_1}{r'}\right)dt'^2 - \left(1 + \frac{c^2+v^2}{c^2-v^2}\frac{\alpha_1}{r'}\right)dx'^2 - \left(1 + \frac{\alpha_1}{r'}\right)(dy'^2 + dz'^2) \tag{5.7}$$

at $\alpha_1 = 2\alpha$.

To search for the acceleration of a material particle at rest in the reference frame K' the equations of geodesics

$$\frac{du^i}{ds} + \Gamma^i_{kl} u^k u^l = 0, \tag{5.8}$$

are used with Christoffel symbols $\Gamma^l_{ij} = \frac{1}{2} g^{lk}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)$. For spatial coordinates with indices $k = 2,3,4$ they turn out to be

$$\frac{du^i}{ds} = \frac{1}{2} g^{kk} \frac{\partial g_{11}}{\partial x^k}\left(u^1\right)^2 \tag{5.9}$$

A special case of a system consisting of two moving bodies

Multiplied by a coefficient $c^2 m$, the right part of this expression will coincide with the gravitational force (3.4), since a stationary particle does not transfer momentum (2.11) to the gravitational field:

$$\frac{d\vec{p}^k}{ds} = 0. \tag{5.10}$$

Equations (5.9), disregarding small quantities of a larger order, yield coordinate accelerations

$$\ddot{x}' = -\frac{1}{2}\frac{c^2 x'}{\sqrt{1-\frac{v^2}{c^2}}}\frac{c^2+v^2}{c^2-v^2}\frac{\alpha_1}{\overline{r}'^3},\ \ddot{y}' = -\frac{1}{2}c^2 y'\frac{c^2+v^2}{c^2-v^2}\frac{\alpha_1}{\overline{r}'^3},\ \ddot{z}' = -\frac{1}{2}c^2 z'\frac{c^2+v^2}{c^2-v^2}\frac{\alpha_1}{\overline{r}'^3}. \tag{5.11}$$

If the spatial radius vector of the particle is perpendicular to the line of motion of the bodies ($x' = 0$), the result corresponds to Newtonian gravity with an active gravitational mass of a material particle

$$M_1^g = M_1\frac{c^2+v^2}{c^2-v^2} \tag{5.12}$$

with $M_1 = 2M$. At $v = V$, this formula is identical to the relation between the rest mass of the particle and its passive gravitational mass (4.26). The presence of the Lorentz factor as a coefficient in acceleration along the coordinate x' is because the movement of the particle is considered in the reference frame, relative to which the sources of gravity move along this coordinate.

PESI for the photon motion

To determine the dynamics of a photon in a gravitational field, we will use PESI [5-8]. Interval in pseudo-Riemannian space-time with metric coefficients $\tilde{g}_{ij}$:

$$ds^2 = \tilde{g}_{ij} dx^i dx^j \tag{6.1}$$

after substitutions

$$\tilde{g}_{11} = \rho^2 g_{11}, \ \tilde{g}_{1p} = \rho g_{1p}, \ \tilde{g}_{pq} = g_{pq} \tag{6.2}$$

at $p, q = 2,3,4$ is rewritten as

$$ds^2 = \rho^2 g_{11} dx^{12} + 2\rho g_{1p} dx^1 dx^p + g_{pq} dx^p dx^q \, . \tag{6.3}$$

The condition $ds = 0$ corresponds to the motion of light. With $g_{11} \neq 0$, the variable ρ is given by the expression

$$\rho = \frac{-g_{1p} u^p + \sigma \sqrt{(g_{1p} g_{1q} - g_{11} g_{pq}) u^p u^q}}{g_{11} u^1}, \tag{6.4}$$

where σ take the values ± 1 and 4-velocities u^i are determined provided that μ is an affine parameter. Further, we will consider variations near $\rho = 1$, to which the equality $\tilde{g}_{ij} = g_{ij}$ corresponds. If $g_{11} = 0$ and condition $g_{1p} \neq 0$ is satisfied for at least one p, then it turns out

$$\rho = -\frac{g_{pq} u^p u^q}{2 g_{1k} u^1 u^k}, \tag{6.5}$$

where k takes on the values 2,3,4.

PESI for the photon motion

The Lagrangian of a freely moving particle is chosen as

$$L = -\rho \,. \tag{6.6}$$

For both values (6.4), (6.5), the covariant generalized momenta (2.2) and forces (2.3) take the form

$$p_\lambda = \frac{u_\lambda}{u^1 u_1}\,, \tag{6.7}$$

$$F_\lambda = \frac{1}{2u_1 u^1}\frac{\partial g_{ij}}{\partial x^\lambda} u^i u^j \,. \tag{6.8}$$

The chosen Lagrangian corresponds to the ratio

$$\rho = u^\lambda \frac{\partial L}{\partial u^\lambda} - L \tag{6.9}$$

being the integral of motion [11] and, accordingly, ρ will be the energy of the system combining the light-like particle and the gravitational field given by the metric (2.1).

The equations of motion are found from Hamilton's principle of stationary action (2.4), which, in view of (6.6), can be written in the form

$$S = -\int_{\mu_0}^{\mu_1} \rho \, d\mu \;. \tag{6.10}$$

The energy ρ is non-zero, its variations leave the interval light-like. The equations of motion will be Euler-Lagrange equations (2.5).

PESI for the photon motion

The contravariant vector of generalized momenta is written as

$$p^{\lambda} = \frac{1}{u^1 u_1} u^{\lambda}. \tag{6.11}$$

Physical energy and momenta of photon with frequency ν in Minkowski space-time with affine parameter

$$\mu = ct \tag{6.12}$$

form contravariant 4-vector of momenta $\pi^i = (h\nu / c) u^i$, where h is the Planck constant. For arbitrary affine parameter it is rewritten as

$$\pi^i = \frac{h\nu}{c} \frac{u^i}{u^1}. \tag{6.13}$$

And in pseudo-Riemannian space-time similar energy and momenta of the photon will be put in line with the components of the contravariant vector of momenta. A certain fixed value of the photon's frequency ν_0 is given by the corresponding equality

$$\nu = \frac{\nu_0}{u_1}. \tag{6.14}$$

Comparing expressions for p^{λ} (6.11) and π^i (6.13), we obtain

$$\pi^i = \frac{h\nu_0}{c} p^i. \tag{6.15}$$

PESI for the photon motion

The Lagrangian (6.6) corresponds to a particle with unit energy. For a photon, it is as follows:

$$L_{ph} = \frac{h\nu_0}{c} L \,. \tag{6.16}$$

In this case, gravitational forces acting on a photon

$$Q^l = h\nu_0 F^l \,. \tag{6.17}$$

are assigned to the components of the associated vector of generalized forces

$$F^k = g^{k\lambda} \frac{1}{2u_1 u^1} \frac{\partial g_{ij}}{\partial x^\lambda} u^i u^j \,. \tag{6.18}$$

Consistency of the PESI for the photon and the generalized Fermat principle

The Fermat principle for a stationary gravitational field [1,12] is formulated as follows:

$$\delta t = \frac{1}{c}\delta\int\frac{1}{g_{11}}\left(dl - g_{1k}dx^k\right) = 0\,, \tag{7.1}$$

where dl is element of spatial distance along the ray:

$$dl^2 = \left(\frac{g_{1p}g_{1q}}{g_{11}} - g_{pq}\right)dx^p dx^q\,. \tag{7.2}$$

Denoting

$$df = \frac{1}{g_{11}}\left(dl - g_{1k}dx^k\right) \tag{7.3}$$

and comparing this expression with (6.4), for $\sigma = 1$ for 3 we write

$$\frac{df}{d\mu} = \rho u^1\,. \tag{7.4}$$

Therefore, variation of integral (7.1) is equivalent to variation

$$S_F = \int_{\mu_0}^{\mu_1}\rho u^1 d\mu\,. \tag{7.5}$$

The generalized Fermat principle [10] extends this approach to non-stationary metrics. It applies the Pontryagin minimum principle from the theory of optimal control. Solutions of the resulting dynamic equations

$$Q = u^1\,, \tag{7.6}$$

$$\frac{d}{d\mu}\left(\frac{\partial Q}{\partial \dot{x}^q}\right) - \frac{\partial Q}{\partial x^q} - \frac{\partial Q}{\partial x^1}\frac{\partial Q}{\partial \dot{x}^q} = 0 \tag{7.7}$$

are isotropic geodesics.

Consistency of the PESI for the photon and the generalized Fermat principle

We prove that these equations are identical to the Euler-Lagrange equations (2.5) for the Lagrangian (6.6). The function Q coincides with the expression for the derivative $df/d\mu$ obtained from equation (7.3), provided that the metric coefficients also depend on time. Therefore, from equation (7.4) follows the expression for energy

$$\rho = \frac{Q}{u^1}. \tag{7.8}$$

In view of (6.11), equations (2.5) for the spatial coordinates yield

$$\frac{1}{u^1}\frac{d}{d\mu}\left(\frac{\partial Q}{\partial u^q}\right) - \frac{1}{\left(u^1\right)^2}\frac{\partial Q}{\partial u^q}\frac{du^1}{d\mu} - \frac{1}{u^1}\frac{\partial Q}{\partial x^q} = 0. \tag{7.9}$$

For the time coordinate ($\lambda = 1$), from the Euler-Lagrange equations in the form (2.6) for generalized momenta (6.7) and forces (6.8), equation

$$\frac{du^1}{d\mu} + \frac{u^1}{2u_1}\frac{\partial g_{ij}}{\partial x^1}u^i u^j = 0. \tag{7.10}$$

follows. Comparing it with following from (2.3), (6.6) and (6.8) relation

$$\frac{\partial \rho}{\partial x^\lambda} = -\frac{1}{2u_1 u^1}\frac{\partial g_{ij}}{\partial x^\lambda}u^i u^j \tag{7.11}$$

in view of (7.8), we obtain

$$\frac{du^1}{d\mu} = \left(u^1\right)^2 \frac{\partial (Q/u^1)}{\partial x^1} = u^1 \frac{\partial Q}{\partial x^1}. \tag{7.12}$$

Substituting this expression into equations (7.9) and multiplying them by u^1 gives equations (7.7). That is, the identity of the equations obtained using the generalized Fermat principle and PESI for a light-like particle is proved. They correspond to the variational principles of classical mechanics. Due to the equivalence of solutions obtained from the first principle to isotropic geodesics, the solutions following from the second principle are also equivalent to them. Compared to Fermat's principle, PESI gives a system that has one more equation. This makes it possible to uniquely determine the affine parameter and the energy-momentum vector of the particle.

General form of expression for force

Equation (2.12) for the force acting on a photon (6.18) takes the form

$$\frac{dp^k}{d\mu} + g^{k\lambda}\frac{\partial g_{\lambda i}}{\partial x^l}u^l p^i = g^{k\lambda}\frac{1}{2u_1u^1}\frac{\partial g_{ij}}{\partial x^\lambda}u^i u^j. \tag{8.1}$$

Let us change the affine parameter

$$d\widehat{\mu} = d\mu \cdot u_1 u^1. \tag{8.2}$$

The expression for momenta (6.11) takes the form

$$p^\lambda = \widehat{u}^\lambda, \tag{8.3}$$

where 4-velocity is defined as $\widehat{u}^\lambda = \dfrac{dx^\lambda}{d\widehat{\mu}}$. Substitution (8.2) into expression (8.1) yields

$$\frac{dp^k}{d\widehat{\mu}} + g^{k\lambda}\frac{\partial g_{\lambda i}}{\partial x^l}\widehat{u}^l p^i = \frac{1}{2}g^{k\lambda}\frac{\partial g_{ij}}{\partial x^\lambda}\widehat{u}^i\widehat{u}^j. \tag{8.4}$$

The right side of this formula coincides with the force (3.4) acting on a material particle of unit energy. It is covariant for linearized metrics.

Photon dynamics in the Schwarzschild field

Let us consider the dynamics of a light-like particle in a static centrally symmetric gravitational field described by the Schwarzschild metric (4.1). Generalized momenta (6.7) for cyclic coordinates t, φ are constant motions

$$B = \frac{cdt}{d\mu}, \tag{9.1}$$

$$C = r^2 \sin^2\theta \frac{d\varphi}{d\mu}\left(1 - \frac{\alpha}{r}\right)^{-1}. \tag{9.2}$$

In view of (6.11) and (6.15), the value of the photon energy

$$E_{ph} = c\pi^1 \tag{9.3}$$

for $B = 1$ is $E_{ph0} = h\nu_0$ away from the center of gravity. Considering the motion in the plane $\theta = \pi / 2$, we obtain the angular component of the 4-velocity vector

$$\frac{d\varphi}{d\mu} = \frac{C}{r^2}\left(1 - \frac{\alpha}{r}\right). \tag{9.4}$$

For isotropic curves ($ds = 0$) from (4.1) we find

$$\frac{dr}{d\mu} = \pm\left[\left(1 - \frac{\alpha}{r}\right)^2 - \left(\frac{C}{r}\right)^2\left(1 - \frac{\alpha}{r}\right)^3\right]^{1/2}. \tag{9.5}$$

The only nonzero component of the associated vector of generalized forces (6.18) is

$$F^2 = -\frac{\alpha}{r^2} + \frac{C^2}{r^3}\left(1 - \frac{\alpha}{r}\right)\left(1 - \frac{\alpha}{2r}\right). \tag{9.6}$$

With radial motion ($C=0$) it is equal to

$$F^2 = -\frac{\alpha}{r^2}, \tag{9.7}$$

coinciding with the doubled force acting on the particle in Newtonian gravity. In view of (6.17), it corresponds to the passive gravitational mass of the photon

$$m_p^{ph} = \frac{2h\nu_0}{c^2}. \tag{9.8}$$

This result is consistent with a thought experiment on "weighing" a photon [13], in which it performs periodic motion in the vertical direction between two horizontal reflecting surfaces.

Considering the non-radial motion, in order to avoid the appearance of a fictitious component of momenta and force due to the sphericity of the coordinate system we use the Schwarzschild metric in rectangular coordinates (4.14). As for a material particle, we will consider the motion in the plane $z=0$ corresponding to the value of the angular coordinate $\varphi=0$ in the spherical frame and look for the force acting on a light-like particle at a point $(ct, x, 0, 0)$. Since the 4-velocities are covariant vectors, then from the solutions of the equations of motion of a light-like particle (9.1), (9.4), (9.5) the components of the 4-velocity vector in a rectangular coordinate system can be accessed using transformations (4.16). In view of (4.12) and (4.17), its nonzero components take the form

$$u^1 = 1, \quad u^2 = \pm\frac{\left(1-\dfrac{\alpha}{4\bar{r}}\right)}{\left(1+\dfrac{\alpha}{4\bar{r}}\right)^3}\left[1-\frac{C^2\left(1-\dfrac{\alpha}{4\bar{r}}\right)^2}{\bar{r}^2\left(1+\dfrac{\alpha}{4\bar{r}}\right)^6}\right]^{1/2}, \quad u^3 = \frac{C\left(1-\dfrac{\alpha}{4\bar{r}}\right)^2}{\bar{r}\left(1+\dfrac{\alpha}{4\bar{r}}\right)^6}. \tag{9.9}$$

Photon dynamics in the Schwarzschild field

Substituting these values and

$$u_1 = \left(\frac{1 - \dfrac{\alpha}{4\bar{r}}}{1 + \dfrac{\alpha}{4\bar{r}}} \right)^2 , \tag{9.10}$$

in (6.18), we find the single non-zero component of the force vector acting on the light-like particle:

$$F^2_{rect} = -\frac{\alpha\left(1 - \dfrac{\alpha}{8\bar{r}}\right)}{\bar{r}^2 \left(1 + \dfrac{\alpha}{4\bar{r}}\right)^5 \left(1 - \dfrac{\alpha}{4\bar{r}}\right)} . \tag{9.11}$$

It is converted to

$$F^2_{rect} = -\frac{\alpha\left(1 - \dfrac{\alpha}{8\bar{r}}\right)}{r^2 \left(1 - \dfrac{\alpha^2}{16\bar{r}^2}\right)} . \tag{9.12}$$

The generalized force acting on a photon does not depend on the direction of its motion. This expression differs from the formula (9.7) corresponding to radial motion in spherical coordinates, which is a consequence of the non-covariance of the vector F^l. However, in the limit of weak gravity ($r \gg \alpha$), these expressions converge asymptotically and give Newton's law of gravitation with a passive gravitational mass of a photon (9.8) equal to twice the mass of a material particle of equivalent energy.

Photon dynamics in the Schwarzschild field

The gravitational field of the electromagnetic radiation flux is determined from the solution of the Einstein equations

$$R^i_j - \frac{1}{2}\delta^i_j R = \chi T^i_j$$

for the electromagnetic field energy-momentum tensor

$$T^{EM}_{ij} = \frac{1}{4} g_{ij} F_{kl} F^{kl} - F^k_i F_{jk} \ ,$$

where F_{ij} is the electromagnetic field tensor. In case of weak gravity, it follows from analysis of acceleration of material particle that active gravitational mass of light beam or light packet is twice as much as similar mass of a rod of equivalent energy [14-16]. However, the gravitational interaction between electromagnetic radiation and material particles differs from the same interaction between photons.

The equality of the active and passive gravitational masses of a photon means the fulfillment of Newton's 3rd law in the gravitational interaction of light-particle and material particles and the laws of conservation of energy and momentum.

Summary

- The dynamics of particles in curvilinear space-time is considered using Lagrange mechanics. A correspondence is established between the physical energy and momentum of a particle, determined from non-gravitational interactions, and the contravariant vector of generalized momenta. The obtained dynamic equations include the rate of change of the energy-momentum vector, the components of which express the energy and momentum acquired by the gravitational field when a particle moves in it. This vector is an analogue of the pseudotensor used in conservation laws in tensor form when considering the dynamics of an individual particle.
- Although the obtained generalized forces are not covariant quantities, in the limit of weak gravity, described by the Schwarzschild metric, they express the Newtonian law of gravity with a passive mass of particles corresponding to the active gravitational mass of moving point bodies and a light beam. The passive gravitational mass of a photon does not depend on the direction of its motion. The same will be true for the passive gravitational mass of a material particle moving along an unrestricted trajectory in Schwarzschild space-time. Coinciding with photon active gravitational mass interacting with a material particle, photon passive gravitational mass is equal to twice the mass of a material particle having an energy equivalent to a photon.
- We have considered a system of two closely spaced identical bodies moving in opposite directions, with low potential energy compared to its kinetic energy. It can be described using a metric obtained by applying Lorentz transformations to the Schwarzschild metric. Its gravitational effect on a material particle depends on the angle between the radius vector and the line of bodies motion.
- Application of PESI and generalized Fermat's principle for a light-like particle in a gravitational field leads to the same solution, which is an isotropic geodesic line. PESI defines a system of equations, which, in comparison with the result of the generalized Fermat's principle, has one more equation. This makes it possible to uniquely identify the energy-momentum vector of the particle.

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Thank you for your attention!